

HERMITE POLYNOMIALS AND QUASI-CLASSICAL ASYMPTOTICS

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ABSTRACT. We study an unorthodox variant of the Berezin-Toeplitz type of quantization scheme, on a reproducing kernel Hilbert space generated by the real Hermite polynomials and work out the associated semi-classical asymptotics.

1. INTRODUCTION

At the heart of most approaches to quantization lies the idea of assigning to functions f (the classical observables) suitable operators T_f (quantum observables) depending on an auxiliary parameter h (the Planck constant) in such a way that as $h \searrow 0$, T_f possesses an appropriate asymptotic behaviour reflecting the “(semi)classical limit” of the quantum system. Typically, the functions f live on a manifold equipped with symplectic structure (the phase space) and the required asymptotic behaviour takes the form of the “correspondence principle”

$$(1) \quad T_f T_g - T_g T_f \approx \frac{ih}{2\pi} T_{\{f,g\}}$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket.

For complex manifolds which are not only symplectic but Kähler, a notable example of such a construction is the Berezin-Toeplitz quantization, first formally introduced in [9], though some ideas go back to Berezin [6] and similar quantization techniques had also been introduced by other authors [2, 23]. Namely, assume for simplicity that the phase space Ω is simply connected, so that the Kähler form ω admits a global real-valued potential Ψ , i.e. $\omega = \partial\bar{\partial}\Psi$. Consider the L^2 space

$$(2) \quad L_h^2 = \{f \text{ measurable on } \Omega : \int_{\Omega} |f|^2 e^{-\Psi/h} \omega^n < \infty\} \quad (h > 0),$$

and let $L_{\text{hol},h}^2$ (the weighted Bergman space) be the subspace in L_h^2 of functions holomorphic on Ω , and $P_h : L_h^2 \rightarrow L_{\text{hol},h}^2$ the orthogonal projection. For a bounded measurable function f on Ω , one then defines the Toeplitz operator T_f on $L_{\text{hol},h}^2$ with symbol f by

$$(3) \quad T_f u = P_h(fu).$$

This is, in fact, an integral operator: more precisely, the space $L_{\text{hol},h}^2$ turns out to be a reproducing kernel Hilbert space [3] possessing a reproducing kernel $K_h(x, y)$, and (3) can be rewritten as

$$(4) \quad T_f u(x) = \int_{\Omega} u(y) f(y) K_h(x, y) e^{-\Psi(y)/h} \omega(y)^n.$$

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When the manifold Ω is not simply connected, one has to assume that the cohomology class of ω is integral, so that there exists a Hermitian line bundle \mathcal{L} with the canonical connection whose curvature form coincides with ω ; and the spaces $L^2_{\text{hol},h}$ (and L^2_h) get replaced by the space of all holomorphic (or all measurable, respectively) square-integrable sections of $\mathcal{L}^{\otimes k}$, $k = \frac{1}{h} = 1, 2, 3, \dots$. In any case, under reasonable technical assumptions on Ω (e.g. for Ω compact [9], or for Ω simply connected strictly-pseudoconvex domain in \mathbb{C}^n with smooth boundary $\partial\Omega$ and $e^{-\Psi}$ vanishing to exactly first order at $\partial\Omega$ [13]), the Toeplitz operators satisfy

$$(5) \quad T_f T_g \approx T_{fg} + h T_{C_1(f,g)} + h^2 T_{C_2(f,g)} + \dots \quad \text{as } h \searrow 0,$$

with some bidifferential operators C_j such that $C_1(f, g) - C_1(g, f) = \frac{i}{2\pi} \{f, g\}$, implying in particular that (1) holds. The asymptotic expansion (5) even holds in the strongest possible sense of operator norms, i.e. the difference of the left-hand side and the sum of the first N terms on the right hand side has norm, as an operator on $L^2_{\text{hol},h}$, bounded by a multiple of h^N as $h \searrow 0$, for all $N = 1, 2, 3, \dots$. Furthermore, the bidifferential operators C_j can be expressed in terms of covariant derivatives, with contractions of the curvature tensor and its covariant derivatives as coefficients, thus encoding various geometric properties of (Ω, ω) in an intriguing way.

The Berezin-Toeplitz *Ansatz* above has subsequently been extended to a number of more general contexts outside the Kähler setting, including e.g. that of harmonic Bergman spaces on some special domains [14] [7] [19], or when spaces of holomorphic functions/sections are replaced by eigenspaces of the Spin^c -Dirac operator on a general symplectic manifold or even orbifold [11] [21] [25] [10], while numerous other developments concerned the properties of the cochains C_j or miscellaneous representation-theoretic aspects of the procedure [20] [24] [16] [12] [26] [27] [5].

The purpose of the present paper is to highlight an operator calculus of a completely different flavour, which nonetheless bears certain resemblance to (1) and (4), and arises in a quite unexpected setting — namely, in connection with orthogonal polynomials. Generically, the situation is the following: as explained above, the Berezin-Toeplitz type of quantization relies on the existence of a certain L^2 -space which contains a reproducing kernel Hilbert space as a subspace; the quantization is effected by the projection operator of this subspace. Alternatively, the reproducing kernel $K(x, y)$ defines a family of vectors $K(\cdot, y)$, $y \in \Omega$ in the reproducing kernel Hilbert space, generally called *coherent states* in the literature, and then (4) shows that the quantization may also be defined in terms of these coherent states. However, the existence of coherent states depends only on the reproducing kernel and not on any ambient L^2 -space and indeed, there have been proposals, some very recent [18, 22], to base both the theory of coherent states and geometric quantization using a positive definite kernel alone. The present paper may be thought of as an extension of this line of thought to Berezin-Toeplitz quantization. What is interesting in our present case is that it is the Hermite polynomials, which in a way define the quantum harmonic oscillator, also define the reproducing kernel of our problem.

To be more specific, let $H_n(x)$ stand for the standard Hermite polynomials (see Section 2 below for the details), and, for $0 < \epsilon < 1$, set

$$(6) \quad K_\epsilon(x, y) = \sum_{n=0}^{\infty} \epsilon^n \|H_n\|^{-2} H_n(x) \overline{H_n(y)}, \quad x, y \in \mathbb{R}.$$

Here $\|H_n\|$ denotes the norm in $L^2(\mathbb{R}, e^{-x^2} dx)$, where the $\{H_n\}$ form an orthogonal basis. Then K_ϵ is a positive-definite function, and, hence, determines uniquely a Hilbert space \mathcal{H}_ϵ of functions on \mathbb{R} for which K_ϵ is the reproducing kernel [3]; this space first appeared in [1] when studying “squeezed” coherent states. (Its definition may perhaps seem a bit artificial at first glance, but so must have seemed (2) when it first came around in Berezin’s papers!) For a (reasonable) function f on \mathbb{R} , set

$$(7) \quad T_f u(x) := \int_{\mathbb{R}} u(y) f(y) K_\epsilon(x, y) e^{-y^2} dy.$$

This certainly resembles the expression (4) for Toeplitz operators, however, note that this time there is no L^2 space around like (2) which would contain \mathcal{H}_ϵ as a closed subspace (in fact, the set $\{f(x)e^{-x^2/2} : f \in \mathcal{H}_\epsilon\}$ is a dense, rather than proper closed, subset of $L^2(\mathbb{R})$), so there is no projection like P_h around and the original definition (3) makes no sense. In particular, there is no reason *a priori* even to expect (7) to be defined, not to say bounded, on some space (whereas with (3) it immediately follows that $\|T_f\|$ is not greater than the norm of the operator of “multiplication by f ” on L^2 , hence $\|T_f\| \leq \|f\|_\infty$). It may therefore come as a bit of a surprise that (7) actually yields, for $f \in L^\infty(\mathbb{R})$, a bounded operator on $L^2(\mathbb{R})$, and, moreover, T_f enjoys a nice asymptotic behaviour as $\epsilon \nearrow 1$, which we will see to correspond, in a very natural sense, to the semiclassical limit $\hbar \searrow 0$ in the original quantization setting.

It should be stressed that the resulting asymptotics are not quite of the form (5) and, in particular, (1) does not hold, so that our results claim no direct physical relevance; on the other hand, the same is true as well for some of the generalizations of the classical Toeplitz calculus mentioned two paragraphs above, while not depriving the latter of their mathematical beauty and relevance. We hope the same to be at least partly true also for our developments here and thus justify their disclosure to a wider audience.

We review the necessary standard material on Hermite polynomials in Section 2. The spaces \mathcal{H}_ϵ are discussed in Section 3, and the basic facts about the operators T_f from (7) in Section 4. The asymptotic behaviour is studied in Section 5. In Section 6 we observe how to recover the standard Berezin-Toeplitz quantization on \mathbb{C} using the Hermite *Ansatz* and an appropriate analogue of the Bargmann transform.

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2. HERMITE POLYNOMIALS

The Hermite polynomials $H_n(x)$, $n = 0, 1, 2, \dots$, are defined by the formula

$$(8) \quad H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

They can also be obtained from the generating function

$$(9) \quad e^{2xz - z^2} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x)$$

and satisfy the orthogonality relations

$$(10) \quad \int_{\mathbb{R}} H_n(x) H_m(x) e^{-x^2} dx = n! 2^n \sqrt{\pi} \delta_{mn}.$$

It follows that the *Hermite functions*

$$(11) \quad h_n(x) := (n!2^n \sqrt{\pi})^{-1/2} H_n(x) e^{-x^2/2},$$

$n = 0, 1, 2, \dots$, form an orthonormal basis in the Hilbert space $L^2(\mathbb{R})$ on the real line.

The representation (9) also leads to the explicit formula

$$(12) \quad H_n(x) = n! \sum_{m=0}^{[n/2]} \frac{(-1)^m (2x)^{n-2m}}{m!(n-2m)!},$$

$[x]$ denoting the integer part of x . From this follows the estimate

$$(13) \quad |H_n(z)| \leq \sqrt{n!2^n} e^{\sqrt{2n}|z|}$$

valid for all complex z .

All this, of course, is quite standard and well-known (see e.g. [4], Chapter 6.1), perhaps with the exception of the last estimate for non-real z ; for completeness, we therefore attach a proof. Observe first of all that

$$(14) \quad \sqrt{\frac{n!}{n^n}} \leq \frac{2^{[n/2] [n/2]!}}{n^{[n/2]}}.$$

Indeed, both for $n = 2k$ even and for $n = 2k+1$ odd, the corresponding inequalities

$$\frac{\sqrt{(2k)!}}{(2k)^k} \leq \frac{2^k k!}{(2k)^k}, \quad \frac{\sqrt{(2k+1)!}}{(2k+1)^{k+\frac{1}{2}}} \leq \frac{2^k k!}{(2k+1)^k}$$

reduce to the elementary estimate

$$(2k)! \leq (2 \cdot 4 \cdot 6 \cdots (2k))^2 = 4^k k!^2.$$

Since

$$\frac{2^m m!}{n^m} = \prod_{j=1}^m \frac{j}{n/2}$$

is a decreasing function of m for $0 \leq m \leq \frac{n}{2}$, it follows from (14) that even

$$\sqrt{\frac{n!}{n^n}} \leq \frac{2^m m!}{n^m}, \quad m = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor.$$

Consequently,

$$\frac{\sqrt{n!}}{m!} 2^{\frac{n}{2}-2m} \leq n^{\frac{n}{2}-m} 2^{\frac{n}{2}-m} = (2n)^{\frac{n-2m}{2}}$$

and

$$\sum_{m=0}^{[n/2]} \frac{\sqrt{n!} 2^{\frac{n}{2}-2m} |z|^{n-2m}}{m!(n-2m)!} \leq \sum_{m=0}^{[n/2]} \frac{(\sqrt{2n}|z|)^{n-2m}}{(n-2m)!} \leq e^{\sqrt{2n}|z|},$$

proving (13).

As a corollary, we also get the estimate

$$(15) \quad |h_n(z)| \leq \pi^{-1/4} e^{\sqrt{2n}|z| - \operatorname{Re} z^2/2}, \quad z \in \mathbb{C},$$

for the corresponding Hermite functions.

The last fact we need to recall is the differential equation

$$H_n''(x) - 2xH_n'(x) + 2nH_n(x) = 0$$

for $H_n(x)$, which translates into another differential equation

$$h_n''(x) + (2n+1-x^2)h_n(x) = 0$$

for the Hermite functions h_n . In other words, the (Schrödinger) operator

$$(16) \quad A := \frac{x^2 - 1}{2}I - \frac{1}{2} \frac{d^2}{dx^2}$$

on $L^2(\mathbb{R})$ satisfies

$$(17) \quad Ah_n = nh_n, \quad n = 0, 1, 2, \dots$$

(that is, $A = \sum_n n \langle \cdot, h_n \rangle h_n$).

3. REPRODUCING KERNEL SPACES

For $0 < \epsilon < 1$, the reproducing kernels

$$(18) \quad K_\epsilon(x, y) = \sum_{n=0}^{\infty} \epsilon^n \frac{H_n(x)H_n(y)}{n!2^n\sqrt{\pi}} = \frac{1}{\sqrt{(1-\epsilon^2)\pi}} e^{-\frac{\epsilon^2}{1-\epsilon^2}(x^2+y^2-\frac{2}{\epsilon}xy)},$$

were introduced in [1]; the second equality is known as Mehler's formula. We denote by \mathcal{H}_ϵ the corresponding reproducing kernel space [3]; that is, \mathcal{H}_ϵ is the completion of linear combinations of the functions $K_\epsilon(\cdot, y)$, $y \in \mathbb{R}$, with respect to the scalar product

$$\left\langle \sum_j a_j K_\epsilon(\cdot, y_j), \sum_k b_k K_\epsilon(\cdot, x_k) \right\rangle = \sum_{j,k} a_j \overline{b_k} K_\epsilon(x_k, y_j).$$

We will also use the Hilbert spaces

$$\tilde{\mathcal{H}}_\epsilon = e^{-x^2/2} \mathcal{H}_\epsilon = \{e^{-x^2/2} f(x) : f \in \mathcal{H}_\epsilon\}$$

corresponding to the reproducing kernel

$$(19) \quad \tilde{K}_\epsilon(x, y) = e^{-(x^2+y^2)/2} K_\epsilon(x, y) = \sum_{n=0}^{\infty} \epsilon^n h_n(x) h_n(y).$$

Since the transition from \mathcal{H}_ϵ to $\tilde{\mathcal{H}}_\epsilon$ involves only the multiplication by $e^{-x^2/2}$, we will state the various facts below usually only for one of these spaces.

The following assertion, though not explicitly stated in [1], is fairly straightforward.

Proposition 1. *One has*

$$(20) \quad \tilde{\mathcal{H}}_\epsilon = \{f(x) = \sum_n f_n h_n(x) : \sum_n \epsilon^{-n} |f_n|^2 < \infty\}$$

with the norm in $\tilde{\mathcal{H}}_\epsilon$ being given by

$$(21) \quad \|f\|_\epsilon^2 = \sum_n \epsilon^{-n} |f_n|^2.$$

Proof. Let us temporarily denote the space on the right-hand side of (20) (with the norm given by (21)) by \mathcal{M}_ϵ . From the equality

$$\sum_n |\epsilon^n h_n(y)|^2 \epsilon^{-n} = \sum_n \epsilon^n |h_n(y)|^2 = \tilde{K}_\epsilon(y, y) = \frac{1}{\sqrt{(1-\epsilon^2)\pi}} e^{\frac{\epsilon-1}{\epsilon+1}x^2} < \infty$$

(cf. (18)), it follows that the function

$$\tilde{K}_{\epsilon,y} := \sum_n \epsilon^n h_n(y) h_n = \tilde{K}_\epsilon(\cdot, y)$$

belongs to \mathcal{M}_ϵ , for any $y \in \mathbb{R}$. Furthermore, for any $f = \sum_n f_n h_n \in \mathcal{H}_\epsilon$,

$$\langle f, \tilde{K}_{\epsilon,y} \rangle_\epsilon = \sum_n \epsilon^{-n} f_n \overline{\epsilon^n h_n(y)} = \sum_n f_n h_n(y) = f(y)$$

(here we have used the fact that h_n is real-valued on \mathbb{R}). Thus \tilde{K}_ϵ is the reproducing kernel for \mathcal{M}_ϵ . Since a reproducing kernel Hilbert space is uniquely determined by its reproducing kernel, $\mathcal{M}_\epsilon = \tilde{\mathcal{H}}_\epsilon$, with equality of norms. \square

The last proposition allows for the following interpretation of the spaces $\tilde{\mathcal{H}}_\epsilon$ and \mathcal{H}_ϵ . Recall that the Sobolev space of order s on \mathbb{R} can be defined as the (completion of the) space of all f ($\in \mathcal{D}(\mathbb{R})$) for which

$$\|f\|_s^2 := \langle (I - \Delta)^s f, f \rangle_{L^2(\mathbb{R})} < \infty.$$

By analogy, one could define ‘‘Hermite-Sobolev’’ spaces $\mathcal{W}^s(\mathbb{R})$ on \mathbb{R} by

$$\|f\|_s^2 := \langle (I + A)^s f, f \rangle_{L^2(\mathbb{R})} < \infty.$$

In view of (17), this is equivalent to

$$\mathcal{W}^s(\mathbb{R}) = \{f = \sum_n f_n h_n : \|f\|_s^2 = \sum_n (n+1)^s |f_n|^2 < \infty\}.$$

Our spaces $\tilde{\mathcal{H}}_\epsilon$ are thus obtained upon replacing $(n+1)^s$ by ϵ^{-n} . Back in the context of the ordinary Laplacian, they are thus analogues of the spaces

$$e^{\epsilon \Delta/2} L^2(\mathbb{R}) = \{f : \langle e^{-\epsilon \Delta} f, f \rangle_{L^2(\mathbb{R})} < \infty\}$$

of solutions at time $t = \frac{\epsilon}{2}$ of the heat equation $\frac{\partial u}{\partial t} = \Delta u$, $u(x, 0) = f(x)$ (‘‘caloric functions’’). More precisely, $\tilde{\mathcal{H}}_\epsilon = e^{-A \log \sqrt{\epsilon}} L^2(\mathbb{R})$ is the space of solutions at time $t = -\frac{1}{2} \log \epsilon$ of the modified heat equation

$$\frac{\partial u}{\partial t} = Au, \quad u = u(x, t), \quad t > 0,$$

with initial condition $u(\cdot, 0) \in L^2(\mathbb{R})$.

We conclude this section by showing that \mathcal{H}_ϵ is actually a space of holomorphic functions, like the weighted Bergman spaces mentioned in the Introduction. (The same is true also for the ordinary spaces of caloric functions.)

Theorem 2. *Each $f \in \mathcal{H}_\epsilon$ extends to an entire function on \mathbb{C} , and \mathcal{H}_ϵ is the space of (the restrictions to \mathbb{R} of) holomorphic functions on \mathbb{C} with reproducing kernel*

$$(22) \quad K_\epsilon(x, y) = \sum_{n=0}^{\infty} \epsilon^n \frac{H_n(x) \overline{H_n(y)}}{n! 2^n \sqrt{\pi}} = \frac{e^{-\frac{\epsilon^2}{1-\epsilon^2}(x^2 + \bar{y}^2 - \frac{2}{\epsilon} x \bar{y})}}{\sqrt{(1-\epsilon^2)\pi}}.$$

Proof. By the preceding proposition, we have

$$(23) \quad f = \sum_n f_n (n! 2^n \sqrt{\pi})^{-1/2} H_n,$$

with

$$\sum_n \epsilon^{-n} |f_n|^2 = \|f\|_{\mathcal{H}_\epsilon}^2 < \infty.$$

Consequently, for any $z \in \mathbb{C}$, we get using the estimate (13)

$$\begin{aligned} \sum_n |f_n H_n(z) (n! 2^n \sqrt{\pi})^{-1/2}| &\leq \sum_n |f_n| \pi^{-1/4} e^{\sqrt{2n}|z|} \\ &\leq \pi^{-1/4} \|f\|_{\mathcal{H}_\epsilon} \left(\sum_n \epsilon^n |e^{\sqrt{2n}|z}|^2 \right)^{1/2}. \end{aligned}$$

Since for any fixed $z \in \mathbb{C}$, the radius of convergence of $\sum_n \epsilon^n e^{2\sqrt{2n}|z|}$, with ϵ as the variable, is 1, the expression in the last parentheses is finite for $0 < \epsilon < 1$. Thus the series (23) converges for any $z \in \mathbb{C}$ (and uniformly on compact subsets). This proves the first part of the theorem, and also shows that

$$f(z) = \langle f, K_{\epsilon, z} \rangle_{\mathcal{H}_\epsilon}, \quad z \in \mathbb{C},$$

with

$$K_{\epsilon, z} := \sum_n \epsilon^n \overline{H_n(z) (n! 2^n \sqrt{\pi})^{-1/2}} (n! 2^n \sqrt{\pi})^{-1/2} H_n,$$

that is,

$$K_{\epsilon, z}(w) = \sum_n \epsilon^n \frac{H_n(w) \overline{H_n(z)}}{n! 2^n \sqrt{\pi}},$$

showing that (22) is indeed the reproducing kernel for \mathcal{H}_ϵ on all of \mathbb{C} . \square

4. TOEPLITZ-TYPE OPERATORS

Drawing inspiration from (4), we define, for a function (“symbol”) f on \mathbb{R} , the “Toeplitz operator” $T_f^{(\epsilon)}$, $0 < \epsilon < 1$, on $L^2(\mathbb{R})$ by

$$(24) \quad T_f^{(\epsilon)} u(x) := \int_{\mathbb{R}} u(y) f(y) K_\epsilon(x, y) e^{-y^2} dy.$$

We will also use the analogous operators

$$(25) \quad \begin{aligned} \tilde{T}_f^{(\epsilon)} u(x) &:= \int_{\mathbb{R}} u(y) f(y) \tilde{K}_\epsilon(x, y) dy \\ &= \int_{\mathbb{R}} u(y) f(y) K_\epsilon(x, y) e^{-\frac{x^2+y^2}{2}} dy \end{aligned}$$

on $L^2(\mathbb{R})$ defined using the kernel \tilde{K}_ϵ instead of K_ϵ . Clearly,

$$(26) \quad \tilde{T}_f^{(\epsilon)} u = \mathbf{e}^{-1/2} T_f^{(\epsilon)} \mathbf{e}^{1/2}$$

where we introduced the notation

$$\mathbf{e}(x) := e^{x^2}.$$

It turns out that the operators $T^{(\epsilon)}$ have a bit nicer expression in terms of the Fourier transform, while $\tilde{T}^{(\epsilon)}$ are a bit nicer from the point of view of the “semiclassical” asymptotics as $\epsilon \nearrow 1$. In view of (26), it is always a simple matter to pass from $T^{(\epsilon)}$ to $\tilde{T}^{(\epsilon)}$ or vice versa.

In the formula (4), the reproducing kernel $K_h(x, y)$ is the integral kernel of the orthogonal projection P_h onto $L^2_{\text{hol}, h}$, i.e. of a bounded operator in the corresponding space L^2_h . On the other hand, for \tilde{K}_ϵ we have no such interpretation, in fact the space $\tilde{\mathcal{H}}_\epsilon$, of which \tilde{K}_ϵ is the reproducing kernel, is dense in $L^2(\mathbb{R})$ (this is immediate from Proposition 1 and the fact that $\{h_n\}_{n=0}^\infty$ is an orthonormal basis of $L^2(\mathbb{R})$). The next two results may therefore seem somewhat surprising.

Theorem 3. *The operators $T_f^{(\epsilon)}$ and $\tilde{T}_f^{(\epsilon)}$ are densely defined for any $f \in C^\infty(\mathbb{R})$. Furthermore, $T_f^{(\epsilon)}$ is bounded for $f \in L^\infty(\mathbb{R})$, with*

$$\|T_f^{(\epsilon)}\| \leq C_\epsilon \|f\|_\infty$$

for some constant C_ϵ depending only on ϵ , $0 < \epsilon < 1$.

Proof. By (18),

$$\begin{aligned} T_f^{(\epsilon)}u(x) &= \int_{\mathbb{R}} (fu)(y) e^{-\frac{\epsilon^2}{1-\epsilon^2}(x^2 - \frac{2}{\epsilon}xy + y^2) - y^2} \frac{dy}{\sqrt{(1-\epsilon^2)\pi}} \\ &= \int_{\mathbb{R}} (fu)(y) e^{-\frac{\epsilon^2}{1-\epsilon^2}(x - \frac{y}{\epsilon})^2} \frac{dy}{\sqrt{(1-\epsilon^2)\pi}} \\ &= \int_{\mathbb{R}} (fu)(\epsilon x - \sqrt{1-\epsilon^2}t) e^{-t^2} \frac{dt}{\sqrt{\pi}} \\ &= (\delta_{\sqrt{1-\epsilon^2}}(fu) * \mathbf{e}^{-1})\left(\frac{\epsilon x}{\sqrt{1-\epsilon^2}}\right), \end{aligned} \tag{27}$$

where we have introduced the dilation operator

$$\delta_r u(x) := u(rx).$$

In other words, introducing also the operator

$$Gu := u * \mathbf{e}^{-1}$$

of convolution with the Gaussian \mathbf{e}^{-1} , we obtain

$$T_f^{(\epsilon)} = \delta_{\epsilon/\sqrt{1-\epsilon^2}} G \delta_{\sqrt{1-\epsilon^2}} M_f, \tag{28}$$

where

$$M_f : u \mapsto fu$$

denotes the operator of “multiplication by f ”. If $f \in C^\infty(\mathbb{R})$ and $u \in \mathcal{D}(\mathbb{R})$, the space of smooth functions on \mathbb{R} with compact support, then $fu \in \mathcal{D} \subset \mathcal{S}$, the Schwartz space on \mathbb{R} . Since dilations map \mathcal{S} into itself while

$$Gf = \left(\hat{f} \frac{\mathbf{e}^{-1/4}}{2\sqrt{\pi}} \right)^\vee \tag{29}$$

(here $\hat{\cdot}$ and $^\vee$ denote the Fourier transform and the inverse Fourier transform, respectively) also maps \mathcal{S} into itself, we conclude that

$$T_f^{(\epsilon)}u \in \mathcal{S} \quad \text{for any } f \in C^\infty(\mathbb{R}) \text{ and } u \in \mathcal{D}(\mathbb{R}).$$

Since \mathcal{D} is dense in L^2 and $\mathcal{S} \subset L^2$, this proves the first part of the theorem for $T^{(\epsilon)}$. The assertion for $\tilde{T}^{(\epsilon)}$ is then immediate from (26) and the fact that $\mathbf{e}^{1/2}\mathcal{D} \subset \mathcal{D}$ and $\mathbf{e}^{-1/2}L^2 \subset L^2$.

The second part follows from (28), because $\|M_f\| \leq \|f\|_\infty$ and

$$\|\delta_{\epsilon/\sqrt{1-\epsilon^2}} G \delta_{\sqrt{1-\epsilon^2}}\| = (4\pi\epsilon)^{-1/2} =: C_\epsilon < \infty$$

by an elementary argument and standard properties of the Fourier transform. \square

Theorem 4. *For $f \in L^\infty$ the operator $\tilde{T}_f^{(\epsilon)}$ is bounded on $L^2(\mathbb{R})$.*

Proof. By (19)

$$\tilde{T}_f^{(\epsilon)} u = \sum_n \epsilon^n \langle fu, h_n \rangle h_n.$$

Thus, for any $0 < \epsilon < 1$,

$$\|\tilde{T}_f^{(\epsilon)} u\|^2 = \sum_n \epsilon^{2n} |\langle fu, h_n \rangle|^2 \leq \sum_n |\langle fu, h_n \rangle|^2 = \|fu\|^2 \leq \|f\|_\infty^2 \|u\|^2,$$

so $\|\tilde{T}_f^{(\epsilon)}\| \leq \|f\|_\infty$. \square

We remark that the same argument as in the last proof also shows that $T_f^{(\epsilon)}$ is bounded, for any $f \in L^\infty$, on the weighted space $L^2(\mathbb{R}, e^{-x^2} dx)$.

5. “SEMICLASSICAL” ASYMPTOTICS

The Parseval identity

$$f = \sum_n \langle f, h_n \rangle h_n, \quad f \in L^2(\mathbb{R}),$$

shows that, at least in the weak sense (i.e. as distributions on $\mathbb{R} \times \mathbb{R}$),

$$(30) \quad \sum_n h_n(x) h_n(y) = \delta(x - y).$$

Thus formally

$$T_f^{(\epsilon)} u = fu \quad \text{for } \epsilon = 1,$$

that is, the operator $T_f^{(\epsilon)}$ reduces just to the multiplication operator M_f on $L^2(\mathbb{R})$ (in the sense explained above) for $\epsilon = 1$. This brings forth naturally the question of the finer description of the behaviour of $T_f^{(\epsilon)}$ as $\epsilon \nearrow 1$, in particular, whether one has any analogue of the “semiclassical limit” formulas like (1) or (5) in the traditional procedures.

The latter asymptotics can be found by the usual Laplace (or stationary phase, or WKB) method, see e.g. Hörmander [17, §7.7]. Namely, assume for simplicity that $f \in C^\infty(\mathbb{R})$ and $u \in \mathcal{D}(\mathbb{R})$. We have seen in (27) that

$$\begin{aligned} T_f^{(\epsilon)} u(x) &= \int_{\mathbb{R}} (fu)(y) e^{-\frac{(y-\epsilon x)^2}{1-\epsilon^2}} \frac{dy}{\sqrt{(1-\epsilon^2)\pi}} \\ &= \int_{\mathbb{R}} (fu)(\epsilon x - \sqrt{1-\epsilon^2}t) e^{-t^2} \frac{dt}{\sqrt{\pi}}. \end{aligned}$$

Let us temporarily write, for the sake of brevity, $fu = F$. Standard estimates used in the stationary phase method show that the integration over y outside a small neighbourhood of x gives an exponentially small contribution as $\epsilon \nearrow 1$, while in the integral over that neighbourhood F can be replaced by its Taylor expansion. Thus we arrive at

$$\begin{aligned} \int_{\mathbb{R}} F(\epsilon x - \sqrt{1-\epsilon^2}t) e^{-t^2} \frac{dt}{\sqrt{\pi}} &\approx \sum_{k=0}^{\infty} \frac{F^{(k)}(x)}{k!} \int_{\mathbb{R}} (\epsilon x - \sqrt{1-\epsilon^2}t)^k e^{-t^2} \frac{dt}{\sqrt{\pi}} \\ &= \sum_{j,l=0}^{\infty} \frac{F^{(j+l)}(x)}{j!l!} (\epsilon-1)^l x^l (-\sqrt{1-\epsilon^2})^j \int_{\mathbb{R}} t^j e^{-t^2} \frac{dt}{\sqrt{\pi}} \\ &= \sum_{k,l=0}^{\infty} \frac{F^{(2k+l)}(x)}{(2k)l!} (\epsilon-1)^l x^l (1-\epsilon^2)^k \frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})} \end{aligned}$$

as $\epsilon \nearrow 1$. Writing $1 - \epsilon^2 = (1 - \epsilon)(2 - (1 - \epsilon))$ and using the binomial theorem to get powers of $(1 - \epsilon)$ only, we finally get

$$(31) \quad T_f^{(\epsilon)} u(x) \approx \sum_{k,l,m=0}^{\infty} (1 - \epsilon)^{k+l+m} \frac{(fu)^{(2k+l)}(x) x^l (-1)^{l+m} 2^{k-m} \binom{k}{m}}{l!k!4^k}$$

as $\epsilon \nearrow 1$. In particular,

$$(32) \quad T_f^{(\epsilon)} u = fu + (1 - \epsilon) \left[\left(\frac{f''}{2} - xf' \right) u + (f' - xf) u' + \frac{f}{2} u'' \right] + O((1 - \epsilon)^2).$$

A similar approach could, of course, be applied also to $\tilde{T}_f^{(\epsilon)}$; however, we proceed to use a different argument, which not only recovers the formula (31) (upon passing from $\tilde{T}^{(\epsilon)}$ to $T^{(\epsilon)}$ via the relation (26)) but is also shorter and applicable in other situations.

Recall the Schrödinger (“number”) operator

$$A = \frac{x^2 - 1}{2} I - \frac{1}{2} \frac{d^2}{dx^2}$$

which is an (unbounded) self-adjoint operator on $L^2(\mathbb{R})$ satisfying $Ah_n = nh_n$, $n = 0, 1, 2, \dots$.

Theorem 5. *We have*

$$\tilde{T}_f^{(\epsilon)} = \epsilon^A M_f,$$

where $\epsilon^A = e^{A \log \epsilon}$ is understood in the sense of the spectral theorem. Consequently, as $\epsilon \nearrow 1$,

$$(33) \quad \tilde{T}_f^{(\epsilon)} u \approx \sum_{k=0}^{\infty} \frac{(\log \epsilon)^k}{k!} A^k (fu).$$

Proof. Let us keep our shorthand $F = fu$, assuming for simplicity that $F \in \mathcal{D}(\mathbb{R})$. Then

$$\begin{aligned} \int_{\mathbb{R}} F(y) \tilde{K}_{\epsilon}(x, y) dy &= \int_{\mathbb{R}} F(y) \sum_n \epsilon^n h_n(x) h_n(y) dy \\ &= \sum_n \epsilon^n \langle F, h_n \rangle h_n(x) \\ &= \sum_n \langle F, h_n \rangle \epsilon^A h_n(x) \\ &= \left(\epsilon^A \sum_n \langle F, h_n \rangle h_n \right)(x) \\ &= (\epsilon^A F)(x) = \sum_k \frac{(\log \epsilon)^k}{k!} (A^k F)(x). \end{aligned}$$

Recalling that $F = fu$ gives the result. \square

Of course, using the familiar series

$$\log \epsilon = - \sum_{j=1}^{\infty} \frac{(1 - \epsilon)^j}{j}$$

one could easily pass in (33) from powers of $\log \epsilon$ to powers of $(1 - \epsilon)$.

The beginning of the asymptotic expansion (33) reads $\tilde{T}_f^{(\epsilon)} u = fu + (1-\epsilon)A(fu) + O((1-\epsilon)^2)$, or

$$(34) \quad \tilde{T}_f^{(\epsilon)} = M_f + (1-\epsilon)AM_f + O((1-\epsilon)^2).$$

Using the similar formulas for g and fg and subtracting, we arrive at

$$(35) \quad \tilde{T}_f^{(\epsilon)} \tilde{T}_g^{(\epsilon)} - \tilde{T}_{fg}^{(\epsilon)} = (1-\epsilon)M_f AM_g + O((1-\epsilon)^2)$$

and

$$(36) \quad \begin{aligned} \tilde{T}_f^{(\epsilon)} \tilde{T}_g^{(\epsilon)} - \tilde{T}_g^{(\epsilon)} \tilde{T}_f^{(\epsilon)} &= (1-\epsilon)(M_f AM_g - M_g AM_f) + O((1-\epsilon)^2) \\ &= \frac{1-\epsilon}{2}(M_f D^2 M_g - M_g D^2 M_f) + O((1-\epsilon)^2) \\ &= (1-\epsilon)(M_{\frac{fg''-gf''}{2}} + M_{fg'-gf'} D) + O((1-\epsilon)^2), \end{aligned}$$

where we introduced the notation

$$Du(x) := \frac{du(x)}{dx}$$

for the differentiation operator on \mathbb{R} . Comparing these formulas with (1) and (5) — the role of the Planck constant being now played by the quantity $1-\epsilon$ — we see that, first of all, the role of the Poisson bracket is now played by the (second-order) expression $\frac{fg''-gf''}{2}$; and, secondly, that in addition to the “Toeplitz” operators $\tilde{T}^{(\epsilon)}$, the differentiation operator D appears too.

For $T^{(\epsilon)}$ instead of $\tilde{T}^{(\epsilon)}$, the formulas (35) and (36) get replaced by

$$T_f T_g - T_{fg} = (1-\epsilon) \left[\left(\frac{fg''}{2} - xfg' \right) I + (fg' - xfg) D + \frac{fg}{2} D^2 \right] + O((1-\epsilon)^2)$$

and

$$T_f T_g - T_g T_f = (1-\epsilon) \left[\left(\frac{fg''-gf''}{2} + xf'g - xfg' \right) I + (fg' - f'g) D \right] + O((1-\epsilon)^2),$$

respectively, and a similar comment applies.

6. BEREZIN-TOEPLITZ QUANTIZATION VIA HERMITE POLYNOMIALS

By virtue of (22), the multiplication operator

$$M : f(z) \mapsto \frac{\sqrt{2\epsilon}}{(1-\epsilon^2)^{1/4} \pi^{1/4}} e^{\frac{\epsilon^2}{1-\epsilon^2} z^2} f(z)$$

maps the space \mathcal{H}_ϵ onto the space of holomorphic functions on \mathbb{C} with reproducing kernel

$$F_\epsilon(z, w) := \frac{2\epsilon}{(1-\epsilon^2)\pi} K_\epsilon(z, w) = \frac{2\epsilon}{(1-\epsilon^2)\pi} e^{\frac{2\epsilon}{(1-\epsilon^2)} z \bar{w}},$$

that is, onto the standard Fock (Segal-Bargmann) space

$$\mathcal{F}_\epsilon = L_{\text{hol}}^2(\mathbb{C}, d\mu_\epsilon)$$

of all entire functions on \mathbb{C} square-integrable with respect to the Gaussian measure

$$d\mu_\epsilon(z) := e^{-2\epsilon|z|^2/(1-\epsilon)} dz,$$

where dz stands for the Lebesgue area measure on \mathbb{C} . This can also be checked directly, using the orthogonality relation

$$(37) \quad \int_{\mathbb{C}} H_n(z) \overline{H_m(z)} e^{-\frac{2\epsilon}{1+\epsilon}x^2 - \frac{2\epsilon}{1-\epsilon}y^2} dx dy = \frac{\sqrt{1-\epsilon^2}}{2\epsilon} n! 2^n \pi \epsilon^{-n} \delta_{mn}, \quad z = x + yi,$$

which can be verified using the generating function for H_n , and which implies that the orthonormal basis $\{\epsilon^{n/2} (n! 2^n \sqrt{\pi})^{-1/2} H_n(z)\}_{n=0}^{\infty}$ of \mathcal{H}_{ϵ} is (taking z complex) also an orthonormal basis in $L^2_{\text{hol}}(\mathbb{C}, \frac{2\epsilon}{\sqrt{(1-\epsilon^2)\pi}} e^{-\frac{2\epsilon}{1+\epsilon}x^2 - \frac{2\epsilon}{1-\epsilon}y^2} dx dy)$; see [1].

Correspondingly,

$$E_n(z) := \frac{\epsilon^{n/2}}{\sqrt{n! 2^n \pi^{1/2}}} \frac{\sqrt{2\epsilon}}{\sqrt[4]{(1-\epsilon^2)\pi}} H_n(z) e^{\epsilon^2 z^2 / (1-\epsilon^2)}, \quad n = 0, 1, 2, \dots,$$

form an orthonormal basis in \mathcal{F}_{ϵ} . The operator

$$(38) \quad V : f \mapsto \sum_n \langle f, h_n \rangle E_n$$

taking each h_n into E_n is thus a unitary map of $L^2(\mathbb{R})$ onto \mathcal{F}_{ϵ} , which is a ‘‘Hermite’’ analogue of the Bargmann transform. Explicitly,

$$(39) \quad Vf(z) = \int_{\mathbb{R}} f(y) \beta(z, y) dy,$$

where

$$(40) \quad \begin{aligned} \beta(z, y) &= \sum_n h_n(y) E_n(z) \\ &= \frac{\sqrt{2\epsilon}}{\sqrt[4]{(1-\epsilon^2)\pi}} e^{\frac{\epsilon^2 z^2}{1-\epsilon^2} - \frac{y^2}{2}} K_{\sqrt{\epsilon}}(z, y) \\ &= \frac{\sqrt{2\epsilon}}{(1-\epsilon^2)^{1/4} (1-\epsilon)^{1/2} \pi^{3/4}} e^{-\frac{\epsilon}{1-\epsilon^2} z^2 - \frac{1+\epsilon}{2(1-\epsilon)} y^2 + \frac{2\sqrt{\epsilon}}{1-\epsilon} zy}. \end{aligned}$$

Using the isomorphism V , one can transfer operators on \mathcal{F}_{ϵ} into those on $L^2(\mathbb{R})$. This applies, in particular, also to the Toeplitz operators T_{ϕ} , $\phi \in L^{\infty}(\mathbb{C})$, on \mathcal{F}_{ϵ} , recalled in the Introduction. From the definition

$$\langle T_{\phi} f, g \rangle_{\mathcal{F}_{\epsilon}} = \int_{\mathbb{C}} \phi f \bar{g} d\mu_{\epsilon}, \quad f, g \in \mathcal{F}_{\epsilon},$$

using (39) one obtains for the transferred operator $V^* T_{\phi} V$ on $L^2(\mathbb{R})$

$$(41) \quad V^* T_{\phi} V f(x) = \int_{\mathbb{R}} f(y) k_{\phi}(x, y) dy$$

where

$$k_{\phi}(x, y) = \int_{\mathbb{C}} \beta(z, y) \overline{\beta(z, x)} \phi(z) d\mu_{\epsilon}(z).$$

Recall that the Weyl operator on $L^2(\mathbb{R})$ with symbol $a(x, \xi)$, $x, \xi \in \mathbb{R}$, is defined by

$$W_a f(x) = \int_{\mathbb{R}} \int_{\mathbb{R}} a\left(\frac{x+y}{2}, \xi\right) e^{i(x-y)\xi} f(y) dy \frac{d\xi}{2\pi},$$

where the right-hand side exists as a convergent integral for, say, a and f in the Schwarz space, and in general extends to be well-defined as an oscillatory integral

for more general functions or even distributions f on \mathbb{R} and a on \mathbb{R}^2 ; see e.g. [15]. Performing the ξ integration yields

$$(42) \quad W_a f(x) = \int_{\mathbb{R}} \check{a}\left(\frac{x+y}{2}, x-y\right) f(y) dy,$$

where $\check{\cdot}$ denotes the inverse Fourier transform with respect to the second variable.

Theorem 6. *We have $V^* T_\phi V = W_a$, where*

$$(43) \quad a(x, \xi) = \left(e^{\frac{1-\epsilon^2}{16\epsilon} \Delta} \phi \right) \left(\frac{1+\epsilon}{2\sqrt{\epsilon}} x - \frac{1-\epsilon}{2\sqrt{\epsilon}} i\xi \right), \quad x, \xi \in \mathbb{R}.$$

Here $e^{t\Delta}$, $t > 0$, denotes the standard heat solution operator

$$e^{t\Delta} \phi(w) = \frac{1}{4\pi t} \int_{\mathbb{C}} \phi(z) e^{-|z-w|^2/(4t)} dt.$$

Proof. Comparing (41) and (42) we see that $V^* T_\phi V = W_a$ where $\check{a}(\frac{x+y}{2}, x-y) = k_\phi(x, y)$, or $\check{a}(s, r) = k_\phi(s + \frac{r}{2}, s - \frac{r}{2})$, or

$$a(s, \eta) = \int_{\mathbb{R}} \int_{\mathbb{C}} e^{-ir\eta} \beta(z, s - \frac{r}{2}) \overline{\beta(z, s + \frac{r}{2})} \phi(z) d\mu_\epsilon(z) dr.$$

Substituting (40) for β and carrying out the r integration yields after some calculations

$$a(s, \eta) = \frac{4\epsilon}{(1-\epsilon^2)\pi} \int_{\mathbb{C}} \phi(z) e^{-\frac{[(1+\epsilon)s-2\sqrt{\epsilon}z_1]^2 + [(1-\epsilon)\eta+2\sqrt{\epsilon}z_2]^2}{1-\epsilon^2}} dz, \quad z = z_1 + iz_2,$$

which is (43). \square

Using the standard properties of the Weyl calculus and the last theorem, it is possible to recover the semi-classical asymptotics (5), mentioned in the Introduction, for the Toeplitz operators on the Fock space \mathcal{F}_ϵ . Namely, assume that a symbol a lies in the Shubin (or Grossmann-Loupas-Stein) class $GLS^m(\mathbb{R}^2)$, $m \leq 0$, that is,

$$\sup_{x, \xi \in \mathbb{R}} \frac{|\partial_x^j \partial_\xi^k a(x, \xi)|}{(1 + |x| + |\xi|)^{m-j-k}} < \infty \quad \forall j, k = 0, 1, 2, \dots,$$

and let similarly $b \in GLS^n$, $n \leq 0$. Then it is known that $W_a W_b = W_c$ for a unique $c \in GLS^{m+n}$, and furthermore $c =: a \# b$ has asymptotic expansion

$$(44) \quad (a \# b)(x, \xi) \sim \sum_{k=0}^{\infty} \frac{(i/2)^k}{k!} (\partial_x \partial_\eta - \partial_\xi \partial_y)^k a(x, \xi) b(y, \eta) \Big|_{y=x, \eta=\xi},$$

where “ \sim ” means that the left-hand side differs from the partial sum of the first N terms on the right-hand side by an element from GLS^{m+n-2N} , for all $N = 0, 1, 2, \dots$. Also, for $a \in GLS^m$, $m \leq 0$, and any $t > 0$, one has

$$(45) \quad e^{t\Delta} a \sim \sum_{k=0}^{\infty} \frac{t^k}{k!} \Delta^k a;$$

see e.g. [8, Theorem 3.1 and §7.4]. In particular, this holds for $t = \frac{1-\epsilon^2}{16\epsilon}$; note that then the last formula, in addition to holding in the same sense as in (44) above, at the same time also represents an asymptotic expansion of $e^{t\Delta} a$ as $\epsilon \nearrow 1$ in increasing powers of $(1-\epsilon)$. Introducing momentarily the shorthands

$$\tau_\epsilon := e^{\frac{1-\epsilon^2}{16\epsilon} \Delta}, \quad \kappa_\epsilon \phi(x, \xi) := \phi\left(\frac{1+\epsilon}{2\sqrt{\epsilon}} x, -\frac{1-\epsilon}{2\sqrt{\epsilon}} \xi\right),$$

we thus get for any $\phi \in GLS^m$, $\psi \in GLS^n$, $m, n \leq 0$,

$$V^*(T_\phi T_\psi - T_{\phi\psi})V = W_a$$

where

$$(46) \quad \begin{aligned} a &= (\kappa_\epsilon \tau_\epsilon \phi) \# (\kappa_\epsilon \tau_\epsilon \psi) - \kappa_\epsilon \tau_\epsilon (\phi\psi) \\ &\sim \sum_{j,k,l=0}^{\infty} \frac{(i/2)^k t^{j+l}}{j!k!l!} (\partial_{x,\phi} \partial_{\xi,\psi} - \partial_{\xi,\phi} \partial_{x,\psi})^k (\kappa_\epsilon \Delta^j \phi) (\kappa_\epsilon \Delta^l \psi) \\ &\quad - \sum_{k=0}^{\infty} \frac{t^k}{k!} \kappa_\epsilon \Delta^k (\phi\psi), \end{aligned}$$

where the subscripts in $\partial_{x,\phi}$, $\partial_{\xi,\psi}$, $\partial_{\xi,\phi}$, $\partial_{x,\psi}$ indicate which of the functions ∂_x or ∂_ξ applies to, and $t = \frac{1-\epsilon^2}{16\epsilon}$. Observe that each ∂_ξ picks from κ_ϵ a factor of $(1-\epsilon)$, so that the last “ \sim ” again, in addition to holding in the same sense as in (44), is also an asymptotic expansion in descending powers of $(1-\epsilon)$ as $\epsilon \nearrow 1$. (Note that κ_ϵ evidently maps each GLS^m into itself.)

Since $\kappa_\epsilon(\phi\psi) = (\kappa_\epsilon \phi)(\kappa_\epsilon \psi)$, the top order terms in the two sums in (46) cancel out. The terms with $j+k+l=1$ in the first sum and the term $k=1$ of the second sum combine into

$$\begin{aligned} &t\kappa_\epsilon \Delta \phi \cdot \kappa_\epsilon \psi + t\kappa_\epsilon \phi \cdot \kappa_\epsilon \Delta \psi + \frac{i}{2}(\partial_x \kappa_\epsilon \phi \cdot \partial_\xi \kappa_\epsilon \psi - \partial_\xi \kappa_\epsilon \phi \cdot \partial_x \kappa_\epsilon \psi) - t\kappa_\epsilon \Delta(\phi\psi) \\ &= -2t\kappa_\epsilon [(\partial_x \phi - i\partial_\xi \phi)(\partial_x \psi + i\partial_\xi \psi)]. \end{aligned}$$

Thus, appealing one more time to (45),

$$a = -2t\tau_\epsilon \kappa_\epsilon [(\partial_x \phi - i\partial_\xi \phi)(\partial_x \psi + i\partial_\xi \psi)] + b$$

where $b \in GLS^{m+n-4}$ and also $b = O(t^2)$ as $t \searrow 0$, i.e. $\epsilon \nearrow 1$. Back on the level of T_ϕ , this amount to

$$T_\phi T_\psi - T_{\phi\psi} = -2tT_{(\partial_x \phi - i\partial_\xi \phi)(\partial_x \psi + i\partial_\xi \psi)} + O(t^2),$$

and upon interchanging ϕ, ψ and subtracting,

$$T_\phi T_\psi - T_\psi T_\phi = \frac{i\hbar}{2\pi} T_{\{f,g\}} + O(\hbar^2)$$

with the Poisson bracket $\{\phi, \psi\} = \partial_\xi \phi \partial_x \psi - \partial_x \phi \partial_\xi \psi$ and Planck's constant

$$\hbar = \frac{1-\epsilon^2}{2\epsilon} \pi,$$

thus recovering (1).

Using the further terms in (46), it is plain how to recover the complete semiclassical expansion (5) as well.

We conclude by remarking that analogously to (38), we also have the unitary map

$$U : f \mapsto \sum_n \left(\frac{2\epsilon}{1-\epsilon} \right)^{(n+1)/2} \left\langle f, \frac{z^n}{\sqrt{n!\pi}} \right\rangle E_n(z)$$

in \mathcal{F}_ϵ sending the standard monomial orthonormal basis $\{(\frac{2\epsilon}{1-\epsilon})^{(n+1)/2} (n!\pi)^{-1/2} z^n\}_n$ into E_n ; thus VU^* is the usual Bargmann transform of $L^2(\mathbb{R})$ onto \mathcal{F}_ϵ . Explicitly,

$$Uf(w) = \frac{2\epsilon}{(1-\epsilon^2)^{3/4}\pi} \int_{\mathbb{C}} f(z) e^{\frac{2\epsilon w \bar{z}}{\sqrt{1-\epsilon^2}} - \frac{\epsilon^2}{1-\epsilon^2} (\bar{z}^2 - w^2)} d\mu_\epsilon(w),$$

which can also be written, using the reproducing kernel property,

$$U = T_{1/\psi} \delta_{1/\sqrt{1-\epsilon^2}} T_{\psi}^*, \quad \psi(z) = e^{-\epsilon^2 z^2 / (1-\epsilon^2)},$$

as a product of two Toeplitz operators and a dilation.

Let us conclude by making a conjecture. The lack of an obvious physical interpretation for the results obtained by our “semiclassical analysis” above, might be a reflection of the fact that an underlying localization property of the quantized system is absent here. As is well known, when the reproducing kernel is a subspace of an L^2 -space, there exists a family of localization operators and a positive operator valued measure which define the localization properties of the quantum system in Ω . In the absence of such an ambient space, no such measure is available and hence no obvious sense in which the quantum system is localized in Ω . In any case, the authors find the application of orthogonal polynomials to the construction of the associated reproducing kernel spaces and operators on them a rather charming figment of complex analysis, and hope very much to have at least partly conveyed this feeling to the reader as well.

REFERENCES

- [1] S. T. Ali, K. Gorska, A. Horzela, F.H. Szafraniec: *Squeezed states and Hermite polynomials in a complex variable*, arXiv:1308.4730, J. Math. Phys., to appear.
- [2] S. T. Ali, H.-D. Doebner: *Ordering problem in quantum mechanics: Prime quantization and its physical interpretation*, Phys. Rev. A **41** (1990), 1199–1210.
- [3] N. Aronszajn: *Theory of reproducing kernels*, Trans. Amer. Math. Soc. **68** (1950), 337–404.
- [4] G.E. Andrews, R. Askey, R. Roy, *Special functions*, Cambridge University Press, Cambridge, 1999.
- [5] J. Arazy, H. Upmeyer: *Covariant symbolic calculi on real symmetric domains*, Singular Integral Operators, Factorization and Applications, Oper. Theory Adv. Appl., vol. 142, Birkhäuser, Basel, 2003, pp. 1–27.
- [6] F.A. Berezin: *Quantization*, Math. USSR Izvestiya **8** (1974), 1109–1163.
- [7] P. Blaschke: *Berezin transform on harmonic Bergman spaces on the unit ball in \mathbb{R}^n* , preprint, 2012.
- [8] H. Bommier-Hato, M. Engliš, E.-H. Youssfi: *Dixmier trace and the Fock space*, Bull. Sci. Math., to appear (<http://dx.doi.org/10.1016/j.bulsci.2013.04.009>).
- [9] M. Bordemann, E. Meinrenken, M. Schlichenmaier: *Toeplitz quantization of Kähler manifolds and $gl(n)$, $n \rightarrow \infty$ limits*, Comm. Math. Phys. **165** (1994), 281–296.
- [10] D. Borthwick, A. Uribe: *Nearly Kählerian embeddings of symplectic manifolds*, Asian J. Math. **4** (2000), 599–620.
- [11] X. Dai, K. Liu and X. Ma, *On the asymptotic expansion of Bergman kernel*, J. Differential Geom. **72** (2006), 1–41.
- [12] M. Douglas, S. Klevtsov, *Bergman kernel from path integral*, Comm. Math. Phys. **293** (2010), 205–230.
- [13] M. Engliš: *Weighted Bergman kernels and quantization*, Comm. Math. Phys. **227** (2002), 211–241.
- [14] M. Engliš: *Berezin transform on the harmonic Fock space*, J. Math. Anal. Appl. **367** (2010), 75–97.
- [15] G.B. Folland, *Harmonic analysis in phase space*, Princeton University Press, 1989.
- [16] N. Gammelgaard: *A universal formula for deformation quantization on Kähler manifolds*, arXiv:1005.2094.
- [17] L. Hörmander: *The analysis of linear partial differential operators, vol. I*, Grundlehren der mathematischen Wissenschaften, vol. 256, Springer-Verlag, Berlin - Heidelberg - New York - Tokyo, 1985.
- [18] A. Horzela, F.H. Szafraniec: *A measure-free approach to coherent states*, J. Phys. A: Math. Theor. **45** (2012), 244018 doi:10.1088/1751-8113/45/24/244018.
- [19] J. Jahn: *On the asymptotic expansion of Berezin transform on the half-space*, J. Math. Anal. Appl. **405** (2013), 720–730.

- [20] A.V. Karabegov, M. Schlichenmaier: *Identification of Berezin-Toeplitz deformation quantization*, J. reine angew. Math. **540** (2001), 49–76.
- [21] X. Ma, G. Marinescu: *The first coefficients of the asymptotic expansion of the Bergman kernel of the spin^c Dirac operator*, Internat. J. Math. **17** (2006), 737–759.
- [22] A. Odziejewicz, M. Horowski: *Positive kernels and quantization*, J. Geom. Physics **63** (2013), 80–98.
- [23] E. Prugovečki: *Consistent formulation of relativistic dynamics for massive spin-zero particles in external fields*, Phys. Rev. D **18** (1978), 3655–3673.
- [24] N. Reshetikhin, L. Takhtajan: *Deformation quantization of Kähler manifolds*, L.D. Faddeev’s Seminar on Mathematical Physics, Amer. Math. Soc. Transl. Ser. 2, Vol. 201, AMS, Providence, 2000, pp. 257–276.
- [25] B. Shiffman, S. Zelditch: *Asymptotics of almost holomorphic sections of ample line bundles on symplectic manifolds*, J. reine Angew. Math. **544** (2002), 181–222.
- [26] H. Xu: *An explicit formula for the Berezin star product*, Lett. Math. Phys. **101** (2012), 239–264.
- [27] H. Xu: *On a graph theoretic formula of Gammelgaard for Berezin-Toeplitz quantization*, Lett. Math. Phys. **103** (2013), 145–169.

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